Approximately Maximizing the Broker's Profit in a Two-sided Market

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ABSTRACT

We study how to maximize the broker's (expected) profit in a two-sided market, where he buys items from a set of sellers and resells them to a set of buyers. Each seller has a single item to sell and holds a private value on her item, and each buyer has a valuation function over the bundles of the sellers' items. We consider the Bayesian setting where the agents' values are independently drawn from prior distributions, and aim at designing dominant-strategy incentive-compatible (DSIC) mechanisms that are approximately optimal.

Production-cost markets, where each item has a publicly-known cost to be produced, provide a platform for us to study two-sided markets. Briefly, we show how to covert a mechanism for production-cost markets into a mechanism for the broker, whenever the former satisfies *cost-monotonicity.* This reduction holds even when buyers have general combinatorial valuation functions. When the buyers' valuations are additive, we generalize an existing mechanism to production-cost markets in an approximation-preserving way. We then show that the resulting mechanism is cost-monotone and thus can be converted into an 8-approximation mechanism for two-sided markets.

KEYWORDS

two-sided market; profit maximization; Bayesian mechanism design

1 INTRODUCTION

Two-sided markets are widely studied market structures in economics [17, 18, 20], where a number of buyers and a number of sellers are connected by an intermediary, such as antique markets, used-car markets, and pre-owned house markets. Here each seller has a single item to trade for money and holds a private value for her owned item, while each buyer's private information is a general combinatorial valuation function over the bundles of the sellers' items. A common feature in these situations is that the intermediary keeps the difference between the payments made by the buyers and the payments made to the sellers —that is, the intermediary's *profit.* We call such an intermediary a *broker.* The objective of the broker is to acquire the items from the sellers and resell them to the buyers to maximize her profit. The problem studied in our paper is to design the mechanism in the two-sided market that maximize the broker's profit. For convenience, we refer to the sub-market between the sellers and the broker the *seller-side market* and to the sub-market between the broker and the buyers the *buyer-side market.*

If the broker had all the items, then we would only have the buyer-side market, which is an auction where the broker tries to maximize his revenue. Auctions have been well studied in the literature following the seminal work of Myerson [19]. In the Section 1.2, we will briefly recall the most relevant literature on auctions. If the broker would keep the items, then we only have the seller-side market, which is a procurement game. Budget feasible procurement has been studied by many in the Algorithmic Game Theory literature [6, 8, 12, 24]. The broker wants to maximize his value for the items he buys, subject to a budget constraint.

Although auctions and procurements are closely related to the broker's problem, they cannot be dealt with separately in two-sided markets. Indeed, the difficulty of the broker's problem is to simultaneously and truthfully elicit both the sellers' and the buyers' valuations, so as to generate a good profit.

1.1 Main Results and Techniques

In this paper we assume the values of the sellers and buyers are independently distributed, and we study simple *dominant-strategy incentive compatible* (DSIC) mechanisms. To approximately maximize the (expected) profit of the broker, we first develop a reduction, through which we can directly convert mechanisms for *production-cost markets* into mechanisms for two-sided markets. In a production-cost market, the broker is able to produce all the items, each item has a cost to be produced and the costs are publicly known. Roughly speaking, we say a mechanism for production-cost markets is *cost-monotone* if, when the cost of an item increases, the likelihood that it is sold does not increase. We show that any cost-monotone mechanism for production-cost markets can be converted into a mechanism for two-sided markets via a blackbox approach. This reduction holds for general combinatorial valuation functions of buyers.

Theorem 3.4 (Informal). Any cost-monotone DSIC mechanism that is an α-approximation for production-cost markets, can be converted into a DSIC mechanism that is an α-approximation for two-sided markets.

Next, we use cost-monotonicity as a guideline in constructing concrete mechanisms for two-sided markets. When the buyers have additive valuations, we generalize the duality framework of [5] and the mechanism there to design a cost-monotone mechanism for production-cost markets. Following our reduction, we immediately obtain a mechanism for two-sided markets.

Proc. of the 18th International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2019), N. Agmon, M. E. Taylor, E. Elkind, M. Veloso (eds.), May 2019, Montreal, Canada

https://doi.org/doi

Theorem 4.4 (Informal). When the buyers have additive valuations, there exists a DSIC mechanism for two-sided markets which is an 8-approximation to the optimal profit.

1.2 Related Work

Bayesian auctions have been extensively studied since the seminal work of [19]. For single-parameter settings, Myerson's mechanism is optimal. The problem becomes more complicated in multi-parameter settings [14]. Although the optimal Bayesian incentive-compatible (BIC) mechanisms have been characterized [3, 4], they are too complex to be practical. Also, optimal DSIC mechanisms remain unknown. Thus simple DSIC mechanisms that are approximately optimal have been studied, such as [5, 16, 25]

Two-sided markets are also called double auctions [17], bilateral trading [20] or market intermediation [15] in the literature. Maximizing the broker's profit is an important objective for two-sided market. The seminal paper [20] characterized the optimal mechanism for one seller and one buyer, which is further generalized by [11] to multiple single-parameter sellers and buyers. Unlike our work, [11] studies the Bayesian Incentive Compatible (BIC) mechanisms. DSIC mechanisms are also studied in the literature, but only for some special cases: [15] studies the case of a single buyer and multiple sellers, [1] studies the case of a single seller and multiple buyers, and [13] studies the optimal mechanism when the numbers of sellers and buyers are both constants. Although [7] studies two-sided markets with multiple buyers and multiple sellers, the dealer there has a fixed budget and their mechanism guarantees that the payment to sellers is within the budget. Before our work, it remained unknown how to design a (simple) DSIC mechanism that approximates the optimal profit in multi-parameter settings with general number of sellers and buyers.

Finally, we briefly discuss the efficiency of two-sided markets, which is measured by *gain-from-trade* (GFT), i.e., the total value gained by the buyers minus the value contributed by the sellers. [17] gave the first approximation mechanism for the one seller and one buyer case, and [2] gives approximation mechanisms for multiple buyers with unit demand valuations. Recently, [22] and [23] study the asymptotically efficient mechanisms instead of constant approximations. For maximizing *social welfare*, [9, 10] provide constant-approximation mechanisms.

2 PRELIMINARIES

A two-sided market includes a set M of m sellers, and a set N of n buyers. We consider the setting where each seller j has one item j to sell, so we may refer to items and sellers interchangeably. The total payment made by the buyers is the broker's *revenue*, and his *profit* is the revenue minus the total payment to the sellers.

Each buyer i has valuation $v_i^B : 2^M \to \mathbb{R}^+ \cup \{0\}$ with $v_i^B(\emptyset) = 0$. The function v_i^B is monotone: for any $T \subseteq S \subseteq M$, $v_i^B(T) \le v_i^B(S)$. In our reduction between production-cost and two-sided markets, we consider combinatorial valuations and do not impose any restriction on v_i^B.

Each function v_i^B is independently drawn from a distribution D_i^B over the set of all possible valuation functions, with density function f_i^B and cumulative probability F_i^B. Let $D^B = \times_{i \in N} D_i^B$, $f^B = \times_{i \in N} f_i^B$ and $F^B = \times_{i \in N} F_i^B$. Each seller j's value on her item, $v_j^S \in \mathbb{R}^+ \cup \{0\}$, is independently drawn from a distribution D_j^S, with density function f_j^S and cumulative probability F_j^S. Let $D^S = \times_{j \in M} D_j^S$, $f^S = \times_{j \in M} f_j^S$ and $F^S = \times_{j \in M} F_j^S$. Let the supports of distributions D_i^B and D_j^S be T_i^B and T_j^S, respectively. T_i^B and T_j^S are called the *valuation spaces* of buyer i and seller j. Let $T^B = \times_{i \in N} T_i^B$ and $T^S = \times_{j \in M} T_j^S$. Finally, denote by $\mathcal{I} = (N, M, D^B, D^S)$ a two-sided market instance.

A mechanism $\mathcal{M}$ for two-sided markets is a tuple of four functions represented by (x^B, x^S, p^B, p^S). Given a valuation profile (v^B, v^S),

- $x^B(v^B, v^S) \triangleq (x_i^B(v^B, v^S))_{i \in N}$ is the allocation of the buyers, where $x_i^B(v^B, v^S) = (x_{iA}^B(v^B, v^S))_{A \subseteq M}$ with $x_{iA}^B(v^B, v^S) \in [0, 1]$, representing the probability that buyer i gets the item set A, under valuation profile v^B and v^S. Moreover, $\sum_A x_{iA}^B(v^B, v^S) = 1$.
- $x^S(v^B, v^S) = (x_j^S(v^B, v^S))_{j \in M}$ is the allocation of the sellers with $x_j^S(v^B, v^S) \in [0, 1]$, representing the probability that seller j's item is sold under (v^B, v^S).
- $p^B(v^B, v^S) = (p_i^B(v^B, v^S))_{i \in N}$ is the payment made by the buyers, where $p_i^B(v^B, v^S) \in \mathbb{R}^+ \cup \{0\}$.
- $p^S(v^B, v^S) = (p_j^S(v^B, v^S))_{j \in M}$ is the payment made to the sellers, where $p_j^S(v^B, v^S) \in \mathbb{R}^+ \cup \{0\}$.

A *feasible* mechanism $\mathcal{M}$ is such that

$$\sum_{A \ni j} \sum_{i \in N} x_{iA}^B(v^B, v^S) \le x_j^S(v^B, v^S)$$

for any item $j \in M$ and any valuation profile (v^B, v^S). In principle, the above condition may allow a mechanism to sell an item that it didn't buy or to buy an item without selling it. However, these cases never happen in the mechanisms in this paper.[1] The expected profit $PFT(\mathcal{M}; \mathcal{I})$ of mechanism $\mathcal{M}$ for instance $\mathcal{I}$ is

$$\mathop{\mathbb{E}}_{v^S \sim D^S; v^B \sim D^B} \sum_{i \in N} p_i^B(v^B, v^S) - \sum_{j \in M} p_j^S(v^B, v^S).$$

The utilities of the agents are quasi-linear. That is, for each buyer i, for any valuation subprofile v_{-i}^B of the buyers and any valuation profile v^S of the sellers, when i reports his true valuation function v_i^B, his utility under mechanism $\mathcal{M}$ is

$$u_i^B(v_i^B; \mathcal{M}, v_{-i}^B, v^S) = \sum_{A \subseteq M} x_{iA}^B(v^B, v^S) v_i^B(A) - p_i^B(v^B, v^S).$$

For each seller j, for any valuation subprofile v_{-j}^S and v^B, when j reports her true value v_j^S, her utility is

$$u_j^S(v_j^S; \mathcal{M}, v^B, v_{-j}^S) = p_j^S(v^B, v^S) - v_j^S x_j^S(v^B, v^S).$$

[1] Note that our feasibility constraint only requires "feasible in expectation" which is weaker than ex post feasibility. All of our results still hold if we change the requirement to be ex post feasible.

Mechanism $\mathcal{M}$ is *dominant-strategy incentive-compatible* (DSIC) if: (1) for any buyer i, v_{-i}^B, v^S, and v_i^B, $v_i'^B$,

$$u_i^B(v_i^B;\mathcal{M},v^B,v_{-i}^S) \geq \sum_{A\subseteq M} x_{iA}^B(v_i'^B,v_{-i}^B,v^S)v_i^B(A) - p_i^B(v_i'^B,v_{-i}^B,v^S);$$

and (2) for any seller j, v_{-j}^S, v^B and v_j^S, $v_j'^S$,

$$u_j^S(v_j^S;\mathcal{M},v_{-j}^S,v^B) \geq p_j^S(v^B,v_j'^S,v_{-j}^S) - v_j^S x_j^S(v^B,v_j'^S,v_{-j}^S).$$

Mechanism $\mathcal{M}$ is *individually rational* (IR) if: (1) for any buyer i, v_i^B, v_{-i}^B and v^S, $u_i^B(v_i^B;\mathcal{M},v_{-i}^B,v^S) \geq 0$; and (2) for any seller j, v_j^S, v_{-j}^S and v^B, $u_j^S(v_j^S;\mathcal{M},v_{-j}^S,v^B) \geq 0$.

Mechanism $\mathcal{M}$ is *Bayesian incentive-compatible* (BIC) if (1) for any buyer i and valuation functions v_i^B, $v_i'^B$,

$$u_i^B(v_i^B;\mathcal{M}) \triangleq \mathbb{E}_{v_{-i}^B\sim D_{-i}^B;v^S\sim D^S}\, u_i^B(v_i^B;\mathcal{M},v_{-i}^B,v^S)$$
$$\geq \mathbb{E}_{v_{-i}^B\sim D_{-i}^B;v^S\sim D^S}\Big[\textstyle\sum_{A\subseteq M} x_{iA}^B(v_i'^B,v_{-i}^B,v^S)v_i^B(A) - p_i^B(v_i'^B,v_{-i}^B,v^S)\Big];$$

and (2) for any seller j and values v_j^S, $v_j'^S$,

$$u_j^S(v_j^S;\mathcal{M}) \triangleq \mathop{\mathbb{E}}_{v^B\sim D^B;v_{-j}^S\sim D_{-j}^S} u_j^S(v_j^S;\mathcal{M},v^B,v_{-j}^S) \geq \mathop{\mathbb{E}}_{v^B\sim D^B;v_{-j}^S\sim D_{-j}^S}\Big[p_j^S(v^B,v_j'^S,v_{-j}^S) - v_j^S x_j^S(v^B,v_j'^S,v_{-j}^S)\Big].$$

Mechanism $\mathcal{M}$ is *Bayesian individually rational* (BIR) if (1) for any buyer i and valuation function v_i^B, $u_i^B(v_i^B;\mathcal{M}) \geq 0$; and (2) for any seller j and value v_j^S, $u_j^S(v_j^S;\mathcal{M}) \geq 0$.

Finally, we denote by $OPT(\mathcal{I})$ the (expected) profit generated by the optimal DSIC mechanism for instance $\mathcal{I}$.

A special case of two-sided markets is *production-cost markets*, where the broker can produce the items by himself and each item $j \in M$ has a publicly known production cost $c_j \in \mathbb{R}^+\cup\{0\}$. Therefore we do not need to consider the sellers' incentives. Letting $c \triangleq (c_j)_{j\in M}$, we use $\mathcal{I}^c = (N,M,D^B,c)$ to denote a production-cost market instance and $\mathcal{M}^c = (x^B,p^B)$ a production-cost market mechanism, where the input of x^B and p^B is the buyers' valuation profile. Then the broker's profit is the revenue minus the total production cost $PFT(\mathcal{M}^c;\mathcal{I}^c)$, which is

$$\mathop{\mathbb{E}}_{v^B\sim D^B}\sum_{i\in N}\left(p_i^B(v^B) - \sum_{A\subseteq M}\sum_{j\in A} x_{iA}^B(v^B)c_j\right).$$

Auctions are production-cost markets with cost 0. We use $\mathcal{I}^a = (N,M,D^B)$ to denote an auction instance and $\mathcal{M}^a = (x^B,p^B)$ an mechanism. The expected revenue is

$$PFT(\mathcal{M}^a;\mathcal{I}^a) = \mathbb{E}_{v^B\sim D^B}\sum_{i\in N} p_i^B(v^B).$$

When there is no ambiguity, the superscript B is omitted in auctions and production-cost markets.

In Section 4, we will consider additive valuations for the buyers. In this case, for any buyer i, there exists a valuation vector $(v_{ij}^B)_{j\in M}$ such that $v_{ij}^B = v^B(\{j\})$ is i's value on each item j. Then, v_i^B is *additive* if $v_i^B(A) = \sum_{j\in A} v_{ij}^B$ for any $A \subseteq M$. To simplify the notation, in this case we use v_i^B to denote the vector $(v_{ij}^B)_{j\in M}$ instead of the corresponding function. Each v_{ij}^B is independently drawn from a distribution D_{ij}^B, and $D_i^B = \times_{j\in M} D_{ij}^B$. Finally, when buyers have additive valuations, their allocation is simplified as $x^B(v^B,v^S) \triangleq (x_i^B(v^B,v^S))_{i\in N}$, where $x_i^B(v^B,v^S) = (x_{ij}^B(v^B,v^S))_{j\in M}$ with $x_{ij}^B(v^B,v^S) \in [0,1]$, representing the probability that buyer i gets the item j, when the valuations are v^B and v^S.

3 A REDUCTION FROM TWO-SIDED MARKETS TO PRODUCTION-COST MARKETS

Note that the sellers are single-parameter in the two-sided markets under consideration. Thus, each seller is truthful in a mechanism if and only if the selling probability of her item is non-increasing with respect to her value and the payment to her is the threshold payment, i.e., the highest value such that her item can still be sold. More precisely, for any single-value distribution D with density function f and cumulative probability F, if D is a seller's value distribution, then the virtual value function is $\phi^S(v) = v + \frac{F(v)}{f(v)}$. In addition, if D is not regular then ϕ^S is the ironed virtual value. Following [20], for single-parameter sellers and any DSIC mechanism $\mathcal{M} = (x^S,x^B,p^S,p^B)$, the total payment to the sellers is the virtual social welfare of them, i.e.,

$$\mathop{\mathbb{E}}_{v^S\sim D^S}\sum_{j\in M} p_j^S(v^B,v^S) = \mathop{\mathbb{E}}_{v^S\sim D^S}\sum_{j\in M}\phi_j(v_j^S)x_j^S(v^B,v^S) \quad (1)$$

for any valuation profile v^B of the buyers.

We now show how to convert a mechanism for production-cost markets into a two-sided market's mechanism. The main idea is to use the sellers' virtual values in two-sided markets as costs, and run the mechanism for production-cost markets.

Definition 3.1. A mechanism $\mathcal{M}^c = (x,p)$ for production-cost markets is *cost-monotone* if for any two instances $\mathcal{I}^c = (N,M,D^c,c)$ and $\mathcal{I}'^c = (N,M,D^c,c')$, where c and c' differ only at an item j and $c_j \leq c_j'$, for any buyers' valuation profile $v^c \sim D^c$, the probabilities of item j being sold under the two instances, $x_j \triangleq \sum_{i\in N}\sum_{A\ni j} x_{iA}(v^c;\mathcal{I}^c)$ and $x_j' \triangleq \sum_{i\in N}\sum_{A\ni j} x_{iA}(v^c;\mathcal{I}'^c)$, satisfy $x_j \geq x_j'$.

Reduction. Let $\mathcal{I} = (N,M,D^S,D^B)$ be a two-sided market instance. For any valuation profile v^S of the sellers, denote by $\phi^S(v^S) \triangleq (\phi_j^S(v_j^S))_{j\in M}$ the sellers' virtual-value vector, and let $\mathcal{I}_{\phi^S(v^S)}^c = (N,M,D^B,\phi^S(v^S))$ be a production-cost market instance.

We first show that the optimal profit of the two-sided market is no more than the optimal profit generated by the corresponding production-cost markets in expectation.

Lemma 3.2. *For any two-sided market instance $\mathcal{I} = (N,M,D^B,D^S)$, $OPT(\mathcal{I}) \leq \mathbb{E}_{v^S\sim D^S} OPT(\mathcal{I}_{\phi^S(v^S)}^c)$.*

Proof. It suffices to show that for any DSIC mechanism $\mathcal{M} = (x^S,x^B,p^S,p^B)$ for two-sided markets, there exists a DSIC mechanism $\mathcal{M}^c$ for production-cost markets such that $PFT(\mathcal{M};\mathcal{I}) \leq$

$\mathbb{E}_{v^S\sim D^S}PFT(\mathcal{M}^c;\mathcal{I}^c_{\phi^S(v^S)})$. Indeed, this would imply $PFT(\mathcal{M};\mathcal{I}) \leq \mathbb{E}_{v^S\sim D^S}OPT(\mathcal{I}^c_{\phi^S(v^S)})$ for any $\mathcal{M}$, and thus

$$OPT(\mathcal{I}) \leq \mathbb{E}_{v^S\sim D^S}OPT(\mathcal{I}^c_{\phi^S(v^S)}).$$

Given $\mathcal{M}$ and $\mathcal{I}$, we define mechanism $\mathcal{M}^c = (x^c, p^c)$ as follows. For any instance $\mathcal{I}^c_{\phi^S(v^S)}$, $\mathcal{M}^c$ first computes v^S, the (randomized) *pre-image* of $\phi^S(v^S)$ with respect to D^S. In particular, if for some seller j, the (ironed) virtual value $\phi^S_j(v^S_j)$ corresponds to a value interval in the support of D^S_j, then v^S_j is randomly sampled from D^S_j conditional on it belongs to this interval.

For any reported valuation profile v^B and buyer $i \in N$,

$$x^c_{iA}(v^B) = x^B_{iA}(v^B, v^S)$$

for any $A \subseteq M$, and

$$p^c_i(v^B) = p^B_i(v^B, v^S).$$

It is easy to see that, given any v^S and v^B_{-i}, for any true valuation v^B_i, buyer i has the same utility in $\mathcal{M}^c$ and $\mathcal{M}$ by reporting the same v'^B_i. Thus $\mathcal{M}^c$ is DSIC whenever $\mathcal{M}$ is DSIC. Next, we lower-bound the profit of $\mathcal{M}^c$ for each instance $\mathcal{I}^c_{\phi^S(v^S)}$.

$$\begin{aligned}
& PFT(\mathcal{M}^c;\mathcal{I}^c_{\phi^S(v^S)}) \\
= & \mathop{\mathbb{E}}_{v^B\sim D^B} \sum_{i\in N}\left(p^c_i(v^B) - \sum_{A\subseteq M} x^c_{iA}(v^B)\sum_{j\in A}\phi^S_j(v^S_j)\right) \\
= & \mathop{\mathbb{E}}_{v^B\sim D^B}\mathop{\mathbb{E}}_{v^S\sim D^S|\phi^S(v^S)}\left(\sum_{i\in N} p^B_i(v^B,v^S)\right. \\
& \qquad \left. - \sum_{j\in M}\sum_{i\in N}\sum_{A\ni j} x^B_{iA}(v^B,v^S)\phi^S_j(v^S_j)\right) \\
\geq & \mathop{\mathbb{E}}_{v^B\sim D^B}\mathop{\mathbb{E}}_{v^S\sim D^S|\phi^S(v^S)}\left(\sum_{i\in N} p^B_i(v^B,v^S)\right. \\
& \qquad \left. - \sum_{j\in M}\phi^S_j(v^S_j)x^S_j(v^B,v^S)\right)
\end{aligned}$$

The inequality above is because any feasible mechanism should satisfy $\sum_{i\in N}\sum_{A\ni j} x^B_{iA}(v^B,v^S) \leq x^S_j(v^B,v^S)$ for any $j\in M$ and any valuation profiles v^B, v^S. Thus,

$$\begin{aligned}
& \mathop{\mathbb{E}}_{v^S\sim D^S} PFT(\mathcal{M}^c;\mathcal{I}^c_{\phi^S(v^S)}) \\
= & \mathop{\mathbb{E}}_{\phi^S(v^S)\sim\phi^S(D^S)} PFT(\mathcal{M}^c;\mathcal{I}^c_{\phi^S(v^S)}) \\
\geq & \mathop{\mathbb{E}}_{v^B\sim D^B}\mathop{\mathbb{E}}_{v^S\sim D^S}\left(\sum_{i\in N} p^B_i(v^B,v^S) - \sum_{j\in M}\phi_j(v^S_j)x^S_j(v^B,v^S)\right) \\
= & \mathop{\mathbb{E}}_{v^S\sim D^S, v^B\sim D^B}\left(\sum_{i\in N} p^B_i(v^B,v^S) - \sum_{j\in M} p^S_j(v^B,v^S)\right) \\
= & PFT(\mathcal{M},\mathcal{I}),
\end{aligned}$$

as desired. Here $\phi^S(D^S)$ is the distribution of virtual values induced by D^S, and the second equality is by Equation 1. □

In the following, we show that if a mechanism for production-cost markets is cost-monotone, then it can be converted into a mechanism for two-sided markets.

LEMMA 3.3. *Given any DSIC cost-monotone mechanism $\mathcal{M}^c$ for production-cost markets, there exists a DSIC mechanism $\mathcal{M}$ for two-sided markets such that*

$$PFT(\mathcal{M};\mathcal{I}) = \mathbb{E}_{v^S\sim D^S}PFT(\mathcal{M}^c;\mathcal{I}^c_{\phi^S(v^S)}).$$

PROOF. Given mechanism $\mathcal{M}^c = (x^c, p^c)$, the mechanism $\mathcal{M} = (x^S, x^B, p^S, p^B)$ is defined as follows: $\mathcal{M}$ first collects v^B and v^S reported by the buyers and the sellers, and then run $\mathcal{M}^c$ on the production-cost instance $\mathcal{I}^c_{\phi^S(v^S)} = (N, M, D^B, \phi^S(v^S))$ to obtain $x^c(v^B)$ and $p^c(v^B)$. Then for each buyer i, let

$$x^B_{iA}(v^B,v^S) = x^c_{iA}(v^B)$$

for any $A\subseteq M$ and

$$p^B_i(v^B,v^S) = p^c_i(v^B).$$

For each seller j, let

$$x^S_j(v^B,v^S) = \sum_{i\in N}\sum_{A\ni j} x^c_{iA}(v^S,v^B)$$

and let $p^S_j(v^B,v^S)$ be the threshold payment for j: namely, the highest reported value of seller j such that the probability that item j is bought by the broker is $x^S_j(v^B,v^S)$.

We claim that $\mathcal{M}$ is DSIC. First, the buyers will truthfully report their valuations because $\mathcal{M}^c$ is DSIC and each buyer has the same allocation and payment in $\mathcal{M}$ and $\mathcal{M}^c$. For the sellers, since $\mathcal{M}^c$ is cost-monotone and each (ironed) virtual value function ϕ^S_j is non-decreasing in v^S_j, the allocation x^S_j is non-increasing in v^S_j. As the payments to the sellers are the threshold payments, the sellers are truthful as well.

Next we show that

$$\begin{aligned}
& PFT(\mathcal{M},\mathcal{I}) \\
= & \mathop{\mathbb{E}}_{v^S\sim D^S, v^B\sim D^B}\left(\sum_{i\in N} p^B_i(v^B,v^S) - \sum_{j\in M} p^S_j(v^B,v^S)\right) \\
= & \mathop{\mathbb{E}}_{v^S\sim D^S, v^B\sim D^B}\left(\sum_{i\in N} p^B_i(v^B,v^S) - \sum_{j\in M} x^S_j(v^B,v^S)\phi_j(v^S_j)\right) \\
= & \mathop{\mathbb{E}}_{v^B\sim D^B}\mathop{\mathbb{E}}_{\phi^S(v^S)\sim\phi^S(D^S)}\left(\sum_{i\in N} p^c_i(v^B)\right. \\
& \qquad \left. - \sum_{j\in M}\sum_{i\in N}\sum_{A\ni j} x^c_{iA}(v^B)\phi_j(v^S_j)\right) \\
= & \mathop{\mathbb{E}}_{\phi^S(v^S)\sim\phi^S(D^S)}\mathop{\mathbb{E}}_{v^B\sim D^B}\sum_{i\in N}\left(p^c_i(v^B)\right. \\
& \qquad \left. - \sum_{A\subseteq M}\sum_{j\in A} x^c_{iA}(v^B)\phi_j(v^S_j)\right) \\
= & \mathop{\mathbb{E}}_{v^S\sim D^S} PFT(\mathcal{M}^c;\mathcal{I}^c_{\phi^S(v^S)}).
\end{aligned}$$

Thus Lemma 3.3 holds. □

Combining Lemmas 3.2 and 3.3, we get our first main result.

Theorem 3.4. *Given any DSIC mechanism $\mathcal{M}^c$ for production-cost markets, if $\mathcal{M}^c$ is cost-monotone and is an α-approximation to the optimal profit, then there exists a DSIC mechanism $\mathcal{M}$ for two-sided markets that is an α-approximation to the optimal profit.*

Proof. Mechanism $\mathcal{M}$ is defined as in Lemma 3.3. For any two-sided market instance $\mathcal{I}$,

$$\begin{aligned} PFT(\mathcal{M};\mathcal{I}) &= \mathop{\mathbb{E}}_{v^S\sim D^S} PFT(\mathcal{M}^c;\mathcal{I}^c_{\phi^S(v^S)}) \\ &\geq \frac{1}{\alpha}\mathop{\mathbb{E}}_{v^S\sim D^S} OPT(\mathcal{I}^c_{\phi^S(v^S)}) \geq \frac{1}{\alpha}OPT(\mathcal{I}), \end{aligned}$$

where the equality is by Lemma 3.3 and the last inequality is by Lemma 3.2. □

4 A MECHANISM FOR TWO-SIDED MARKETS WITH ADDITIVE VALUATIONS

4.1 Broker's Profit in Production-Cost Markets

We first design a mechanism $\mathcal{M}_A$ for production-cost markets which is an 8-approximation of the optimal profit. Our mechanism is inspired by the mechanism in [25] and the duality framework in [5] for auctions. In particular, with probability $\frac{3}{4}$, $\mathcal{M}_A$ runs the mechanism of [20] for two-sided markets for each item separately, denoted by $\mathcal{M}_{IT}$. The mechanism of [20] is for a single buyer and a single seller, but can be generalized to multiple buyers and a single seller as shown in [11]. Furthermore, $\mathcal{M}_A$ generalizes the bundling VCG mechanism of [25] to production-cost markets (denoted by $\mathcal{M}_{BVCG}$) and runs it with probability $\frac{1}{4}$.

Essentially, Mechanism $\mathcal{M}_{IT}$ runs a second-price auction on the buyers' virtual values, with a reserve price which is the production cost of the item. As shown in [11, 20], this mechanism is optimal for the broker's profit when the buyers have single-parameter valuations. Mechanism $\mathcal{M}_{BVCG}$ is well studied in auctions [5, 25], and we describe it in Mechanism 1 for production-cost markets $\mathcal{I}^c = (N, M, D, c)$. Essentially, it is a VCG mechanism with per-item reserve prices and per-agent entry fees.

Mechanism 1 $\mathcal{M}_{BVCG}$ for Production-Cost Markets

1: Collect the valuation profile v from the buyers.
2: For any buyer i and item j, let $P_{ij}(v_{-i}) \triangleq \max_{i'\neq i} v_{i'j}$ and $\beta_{ij}(v_{-i}) \triangleq \max\{P_{ij}(v_{-i}), c_j\}$.
3: For any buyer i, set the reserve price for item j to be $\beta_{ij}(v_{-i})$. Set the entry fee $e_i(v_{-i})$ to be the median of the random variable $\sum_{j\in M}(t_{ij} - \beta_{ij}(v_{-i}))^+$, where $t_i = (t_{ij})_{j\in M} \sim D_i$ and $x^+ \triangleq \max\{x, 0\}$ for any $x \in \mathbb{R}$.
4: Each buyer i is considered to accept his entry fee if and only if $\sum_{j\in M}(v_{ij} - \beta_{ij}(v_{-i}))^+ \geq e_i(v_{-i})$.
5: If a buyer i accepts his entry fee, then he gets the set of items j with $v_{ij} \geq \beta_{ij}(v_{-i})$, and his price is $e_i(v_{-i}) + \sum_{j:v_{ij}\geq\beta_{ij}(v_{-i})} \beta_{ij}(v_{-i})$. If i does not accept his entry fee, then he gets no item and pays 0.

It is not hard to see that both $\mathcal{M}_{IT}$ and $\mathcal{M}_{BVCG}$ are DSIC and IR. Indeed, the mechanism of [20] is DSIC and IR, $\mathcal{M}_{IT}$ directly applies it to each item, and the buyers have additive valuations across the items. Moreover, $\mathcal{M}_{BVCG}$ is DSIC and IR with respect to any reserve prices β_{ij} that do not depend on v_{ij}, and Mechanism 1 simply incorporates the production costs into reserve prices.

In Theorem 4.1 use $\mathcal{M}_A$ to upper-bound the optimal profit for any production-cost instance $\mathcal{I}^c = (N, M, D, c)$, with proof provided in the appendix. In fact, this proof is similar to the proof in [5] with modifications to incorporate the production costs into consideration. Note that [2] also adapts the framework of [5] to the 2-sided market. But their goal is to maximize the gain from trade and the buyers have unit-demand valuations.

Theorem 4.1. *When the buyers have additive valuations, Mechanism $\mathcal{M}_A$ is DSIC and is an 8-approximation to the optimal profit for production-cost markets.*

4.2 Converting $\mathcal{M}_A$ to Two-sided Markets

Next we prove the cost-monotonicity for Mechanism $\mathcal{M}_A$. But first, we start with Mechanism $\mathcal{M}_{IT}$.

Lemma 4.2. *$\mathcal{M}_{IT}$ is cost-monotone.*

Proof. For any two production-cost instances $\mathcal{I}^c = (N, M, D, c)$ and $\mathcal{I}'^c = (N, M, D, c')$, where there exists an item $j \in M$ such that $c'_j > c_j$ and $c'_{j'} = c_{j'}$ for any $j' \neq j$, we show that in Mechanism $\mathcal{M}_{IT}$, when buyers' valuation profile is $v \sim D$, if item j is not sold in $\mathcal{I}^c$, then item j is not sold in $\mathcal{I}'^c$. Since all buyers' valuation functions are additive and $\mathcal{M}_{IT}$ sells each item individually, the result of selling one item does not effect any other item. In the mechanism of [20], given the reported valuation profile v, the potential winner of item j is the buyer who has highest virtual value on it, denoted by $i_j = \arg\max_{i\in N} v_{ij}$. If his virtual value $\phi_{i_j j}(v_{i_j j})$ is at least the cost of item j, buyer i_j takes item j. Otherwise, item j is kept unsold. Therefore, if item j is not sold in $\mathcal{I}^c$, then $\phi_{i_j j}(v_{i_j j}) - c_j < 0$ which implies $\phi_{i_j j}(v_{i_j j}) - c'_j < 0$ and item j cannot be sold in $\mathcal{I}'^c$. Thus $\mathcal{M}_{IT}$ satisfies cost-monotonicity. □

Next we show $\mathcal{M}_{BVCG}$ is cost-monotone. Since we need to apply $\mathcal{M}_{BVCG}$ to different instances with different cost vectors c and c', we explicitly write $\beta_{ij}(v_{-i}, c_j)$ and $e_i(v_{-i}, c)$ in Steps 2 and 3 of Mechanism 1.

Lemma 4.3. *$\mathcal{M}_{BVCG}$ is cost-monotone.*

Proof. Similarly, for any two production-cost instances $\mathcal{I}^c = (N, M, D, c)$ and $\mathcal{I}'^c = (N, M, D, c')$, where there exists an item $j \in M$ such that $c'_j > c_j$ and $c'_{j'} = c_{j'}$ for any $j' \neq j$, we show that in Mechanism $\mathcal{M}_{BVCG}$, when buyers' valuation profile is $v \sim D$, if item j is sold in $\mathcal{I}'^c$, then item j is also sold in $\mathcal{I}^c$.

In Mechanism $\mathcal{M}_{BVCG}$, given the valuation profile v, the potential winner of item j is the buyer who has highest value on item j, denoted by $i_j = \arg\max_{i\in N} v_{ij}$. When the cost vector is c, item j is sold to i_j if and only if i_j accepts the entry fee $e_{i_j}(v_{-i}, c)$ and $v_{i_j} - \beta_{i_j j}(v_{-i_j}, c_j) > 0$. Otherwise item j is unsold. Note that given different production cost c', the potential winner of item j remains unchanged.

Note that the entry fee $e_i(v_{-i}, c)$ is selected such that the probability that buyer i accepts it is exactly $\frac{1}{2}$. Let $d_j \triangleq \beta_{i_j j}(v_{-i_j}, c'_j) - \beta_{i_j j}(v_{-i_j}, c_j)$ be the increase of the item reserve in $\mathcal{M}_{BVCG}$ for buyer i_j. Then

$$(v_{i_j j} - \beta_{i_j j}(v_{-i_j}, c'_j))^+ + d_j \geq (v_{i_j j} - \beta_{i_j j}(v_{-i_j}, c_j))^+. \qquad (2)$$

Indeed the equality holds in Inequality 2 if $v_{i_j j} - \beta_{i_j j}(v_{-i_j}, c'_j) \geq 0$.

Let $e'_{i_j} \triangleq e_{i_j}(v_{-i_j}, c) - d_j$. When the cost vector is c, buyer i_j's utility is

$$u_{i_j} = \sum_{k \in M} (v_{i_j k} - \beta_{i_j k}(v_{-i_j}, c_k))^+ - e_{i_j}(v_{-i_j}, c).$$

When the cost is c', if the entry fee is e'_{i_j}, buyer i_j's utility is

$$\begin{aligned}
u'_{i_j}(e'_{i_j}) &= \sum_{k \in M} (v_{i_j k} - \beta_{i_j k}(v_{-i_j}, c'_k))^+ - e'_{i_j} \\
&= \sum_{k \neq j} (v_{i_j k} - \beta_{i_j k}(v_{-i_j}, c_k))^+ + (v_{i_j j} - \beta_{i_j j}(v_{-i_j}, c'_j))^+ - e'_{i_j} \\
&= \sum_{k \neq j} (v_{i_j k} - \beta_{i_j k}(v_{-i_j}, c_k))^+ + (v_{i_j j} - \beta_{i_j j}(v_{-i_j}, c'_j))^+ \\
&\quad - e_{i_j}(v_{-i_j}, c) + d_j \\
&\geq \sum_{k \neq j} (v_{i_j k} - \beta_{i_j k}(v_{-i_j}, c_k))^+ + (v_{i_j j} - \beta_{i_j j}(v_{-i_j}, c_j))^+ \\
&\quad - e_{i_j}(v_{-i_j}, c) \\
&= u_{i_j}.
\end{aligned}$$

The inequality above is by Inequality 2. Moreover, if $v_{i_j j} - \beta_{i_j j}(v_{-i_j}, c'_j) \geq 0$, then

$$u'_{i_j}(e'_{i_j}) = u_{i_j}. \qquad (3)$$

That is for any valuation profile v_{i_j}, buyer i_j's utility under the entry fee e'_{i_j} and the cost vector c' is at least his utility under the entry fee $e_{i_j}(v_{-i_j}, c)$ and the cost vector c. Therefore,

$$\Pr_{t_{i_j} \sim D_{i_j}} [\sum_{j \in M} (t_{i_j j} - \beta_{i_j j}(v_{-i_j}))^+ \geq e'_{i_j}] \geq \frac{1}{2}.$$

Since the real entry fee $e_{i_j}(v_{-i_j}, c')$ is selected to be the median of the random variable $\sum_{j \in M} (t_{i_j j} - \beta_{i_j j}(v_{-i_j}))^+$, we have $e_{i_j}(v_{-i_j}, c') \geq e'_{i_j}$.

Now if under cost vector c', item j is sold, then

$$v_{i_j j} - \beta_{i_j j}(v_{-i_j}, c'_j) \geq 0,$$

and

$$\sum_{k \in M} (v_{i_j k} - \beta_{i_j k}(v_{-i_j}, c'_k))^+ - e_{i_j}(v_{-i_j}, c') \geq 0.$$

Thus under cost vector c, we have

$$v_{i_j j} - \beta_{i_j j}(v_{-i_j}, c_j) \geq 0,$$

and by Equation 3,

$$\begin{aligned}
u_{i_j} &= u'_{i_j}(e'_{i_j}) = \sum_{k \in M} (v_{i_j k} - \beta_{i_j k}(v_{-i_j}, c'_k))^+ - e'_{i_j} \\
&\geq \sum_{k \in M} (v_{i_j k} - \beta_{i_j k}(v_{-i_j}, c'_k))^+ - e_{i_j}(v_{-i_j}, c') \geq 0.
\end{aligned}$$

Therefore, under cost vector c, item j is also sold. That is, $\mathcal{M}_{BVCG}$ is cost-monotone. □

By randomly selecting from $\mathcal{M}_{IT}$ and $\mathcal{M}_{BVCG}$, Mechanism $\mathcal{M}_A$ is still cost-monotone. Therefore, by Theorem 3.4, $\mathcal{M}_A$ can be converted into a mechanism for two-sided markets, which is again an 8-approximation to the optimal profit. Formally, we have the following theorem.

Theorem 4.4. *When the buyers have additive valuations, there exists a DSIC mechanism that is an 8-approximation to the optimal profit for two-sided markets.*

5 CONCLUSION AND OPEN PROBLEMS.

In this paper we provide the first DSIC mechanism that is a constant-approximation to the broker's optimal profit in multi-parameter settings with more than one buyer. We use production-cost markets as a bridge between auctions and two-sided markets, and provide a general reduction from production-cost markets to two-sided markets. How to design DSIC mechanisms for the broker's profit in multi-buyer multi-parameter settings under other valuation functions of the buyers (e.g. unit-demand or sub-additive) is still open, and it would be interesting to understand the role of production-cost markets in those scenarios.

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A PROOF OF THEOREM 2

Similar to [5], we only need to consider the prior distribution D with finite support.

Arbitrarily fix a BIC-BIR mechanism $\mathcal{M} = (x, p)$. We consider $\mathcal{M}$ in its ex ante form. Specifically, for any buyer i with valuation v_i and for any item j, let (a) $x_{ij}(v_i) \triangleq \mathbb{E}_{v_{-i}\sim D_{-i}} x_{ij}(v_i, v_{-i})$ be the probability that buyer i gets item j, over the randomness of the mechanism and v_{-i}; and (b) $p_i(v_i) \triangleq \mathbb{E}_{v_{-i}\sim D_{-i}} p_i(v_i, v_{-i})$ be the expected payment made by i. Then the broker's profit is

$$PFT(\mathcal{M}; \mathcal{I}^c) = \sum_{i\in N}\sum_{v_i\in T_i} D_i(v_i)\left(p_i(v_i) - \sum_j x_{ij}(v_i)c_j\right),$$

where $D_i(v_i)$ is the probability of v_i according to distribution D_i.

Consider the constraints for $\mathcal{M}$. Let $T_i^+ = T_i \cup \{\perp\}$, where "$\perp$" is a special symbol not in the support T_i. Moreover, $x_i(\perp) \triangleq (0, \ldots, 0)$ and $p_i(\perp) \triangleq 0$. Because $\mathcal{M}$ is BIC and BIR, we have

$$x_i(v_i)\cdot v_i - p_i(v_i) \geq x_i(v_i')\cdot v_i - p_i(v_i'), \quad \forall i \in N, v_i \in T_i, v_i' \in T_i^+ \tag{4}$$

In particular, when $v_i' = \perp$, the above constraint restricts the mechanism to be BIR. Therefore, the problem of designing BIC-BIR mechanisms is to maximize $PFT(\mathcal{M}; \mathcal{I}^c)$ subject to Inequality 4.

To upper-bound $PFT(\mathcal{M}; \mathcal{I}^c)$, we first introduce some notations. For any buyer i, item j and valuation sub-profile $v_{-i} \in T_{-i}$, let $P_{ij}(v_{-i}) = \max_{i'\neq i} v_{i'j}$ and $\beta_{ij}(v_{-i}) = \max\{P_{ij}(v_{-i}), c_j\}$, as in Mechanism $\mathcal{M}_{BVCG}$. In Mechanism 2 we generalize the 1-lookahead mechanism [21] to production-cost markets, denoted by $\mathcal{M}_{1LA}$. For any buyer i with valuation v_i and for any item j, let $r_{ij}(v_{-i}) = \max_{p\geq\beta_{ij}(v_{-i})}(p - c_j)\cdot \Pr_{y\sim D_{ij}}[y \geq p]$ and $r_i(v_{-i}) = \sum_{j\in M} r_{ij}(v_{-i})$. Moreover, let $r_i = \mathbb{E}_{v_{-i}\sim D_{-i}} r_i(v_{-i})$ and $r = \sum_i r_i$. Then the profit of $\mathcal{M}_{1LA}$ is exactly r and will be at most $PFT(\mathcal{M}_{IT}; \mathcal{I}^c)$.

Next, let $R_0^{(v_{-i})} = \{v_i \in T_i | v_{ij} \leq \beta_{ij}(v_{-i})$ for all $j\}$ and $R_j^{(v_{-i})} = \{v_i \in T_i | j$ is the smallest index such that $j \in \arg\max_{k\in M}\{v_{ik} - \beta_{ik}(v_{-i})\}$ and $v_{ij} - \beta_{ij}(v_{-i}) > 0\}$. Moreover, let $\mathbb{I}[E]$ be the indicator of an event E. Now we are ready to upper-bound $PFT(\mathcal{M}; \mathcal{I}^c)$. By adopting the duality framework in [5] to production-cost markets, we have the following.

Mechanism 2 $\mathcal{M}_{1LA}$ for Production-Cost Markets

1: Collect the valuation profile v from the buyers.
2: **for** each item j **do**
3: If no buyer has value for j higher than $\beta_{ij}(v_{-i})$, keep j unsold.
4: Otherwise, let i be the highest bidder for j, and
$$\rho_{ij}(v_{-i}) \triangleq \underset{p\geq\beta_{ij}(v_{-i})}{\arg\max}\,(p - c_j)\cdot \Pr_{y\sim D_{ij}}(y \geq p).$$
Sell the item j to buyer i with price $\rho_{ij}(v_{-i})$ if and only if $v_{ij} \geq \rho_{ij}(v_{-i})$.
5: **end for**

Lemma A.1. *For any BIC-BIR mechanism $\mathcal{M} = (x, p)$ and production-cost instance $\mathcal{I}^c = (N, M, D, c)$,*

$$PFT(\mathcal{M}; \mathcal{I}^c) \leq \textsc{Single} + \textsc{Under} + \textsc{Over} + \textsc{Tail} + \textsc{Core},$$

where $\phi_{ij}(v_{ij})$ is Myerson's (ironed) virtual value and

$$\begin{aligned}
\textsc{Single} &= \sum_{i\in N}\sum_{v_i\in T_i}\sum_{j\in M} D_i(v_i)x_{ij}(v_i)(\phi_{ij}(v_{ij}) - c_j),\\
&\quad\cdot \Pr_{t_{-i}\sim D_{-i}}[v_i \in R_j^{(t_{-i})}]\\
\textsc{Under} &= \sum_{i\in N}\sum_{v_i\in T_i}\sum_{j\in M} D_i(v_i)x_{ij}(v_i)\\
&\quad\cdot\Big(\sum_{t_{-i}\in T_{-i}} D_{-i}(t_{-i})(v_{ij} - c_j)\cdot\mathbb{I}[v_{ij} < \beta_{ij}(t_{-i})]\Big),\\
\textsc{Over} &= \sum_{i\in N}\sum_{v_i\in T_i}\sum_{j\in M} D_i(v_i)x_{ij}(v_i)\\
&\quad\cdot\Big(\sum_{t_{-i}\in T_{-i}} D_{-i}(t_{-i})(\beta_{ij}(t_{-i}) - c_j))\\
&\quad\cdot\mathbb{I}[(v_{ij} \geq \beta_{ij}(t_{-i})]\Big),\\
\textsc{Tail} &= \sum_{i\in N}\sum_{j\in M}\sum_{t_{-i}\in T_{-i}} D_{-i}(t_{-i})\\
&\quad\cdot\sum_{v_{ij}>\beta_{ij}(t_{-i})+r_i(t_{-i})} D_{ij}(v_{ij})(v_{ij} - \beta_{ij}(t_{-i}))\\
&\quad\cdot\Pr_{v_{i,-j}\sim D_{i,-j}}[\exists k\neq j, v_{ik} - \beta_{ik}(t_{-i})\\
&\qquad\geq v_{ij} - \beta_{ij}(t_{-i})],\\
\textsc{Core} &= \sum_{i\in N}\sum_{j\in M}\sum_{t_{-i}\in T_{-i}} D_{-i}(t_{-i})\\
&\quad\cdot\sum_{\beta_{ij}(t_{-i})\leq v_{ij}\leq\beta_{ij}(t_{-i})+r_i(t_{-i})} D_{ij}(v_{ij})\\
&\quad\cdot(v_{ij} - \beta_{ij}(t_{-i})).
\end{aligned}$$

In the following, we use our mechanisms to bound the above terms separately. For any production-cost instance $\mathcal{I}^c = (N, M, D, c)$, we define its corresponding single-parameter COPIES instance $\hat{\mathcal{I}}^c = (\hat{N}, M, D, c)$. More precisely, for each buyer $i \in N$, there are m copies for i in set $\hat{N}$, and the j-th copy of i (also denoted by (i, j)) is only interested in item j. Since in $\mathcal{I}^c$, buyers' valuations are additive, $OPT(\hat{\mathcal{I}}^c) = PFT(\mathcal{M}_{IT}; \mathcal{I}^c)$.

In the following, we first use $OPT(\hat{\mathcal{I}}^c)$ to bound Single, Under and Over, which implies that these terms are upper-bounded by $PFT(\mathcal{M}_{IT};\mathcal{I}^c)$.

Lemma A.2. Single $\leq OPT(\hat{\mathcal{I}}^c)$.

Proof. Note that if the allocation rule of $\mathcal{M}$ is feasible for $\mathcal{I}^c$, then it is also feasible for $\hat{\mathcal{I}}^c$. Thus,

$$\begin{aligned}\text{Single} &\leq \sum_{i\in N}\sum_{v_i\in T_i}\sum_{j\in M} D_i(v_i)x_{ij}(v_i)(\phi_{ij}(v_{ij})-c_j)\\ &\leq OPT(\hat{\mathcal{I}}^c).\end{aligned}$$

The first inequality holds because $\Pr_{t_{-i}\sim D_{-i}}[v_i\in R_j^{(t_{-i})}]\leq 1$, and the last inequality holds because the optimal mechanism for single-parameter buyers maximizes the virtual social welfare. □

Next we upper-bound Under. Essentially, the proof of Lemma A.3 shares the same idea with Lemma 15 in [5], but here we have to deal with multiple reserve prices for each item.

Lemma A.3. Under $\leq OPT(\hat{\mathcal{I}}^c)$.

Proof. It suffices show that for any BIC-BIR mechanism $\mathcal{M}$ for $\mathcal{I}^c$, there is a mechanism $\mathcal{M}'$ for the single-parameter instance $\hat{\mathcal{I}}^c$, which achieves profit at least as much as Under.

Given $\mathcal{M}=(x,p)$, we define $\mathcal{M}'=(x',p')$ as follows. Randomly draw $v\sim D$ and run $\mathcal{M}$ on v. If item j is sold to buyer i, let v_{ij} be item j's reserve price in $\mathcal{M}'$. Then for instance $\hat{\mathcal{I}}^c$ with reported valuation profile t, run second-price auction with multiple reserve prices for each item. That is for item j, the highest-bid buyer, say (i_j,j), gets the opportunity to buy it and the price is $\max\{v_{ij},\beta_{i_jj}(t_{-i_j})\}$. If $t_{i_jj}<\max\{v_{ij},\beta_{i_jj}(t_{-i_j})\}$, item j is kept unsold.

Next we compare the profit of $\mathcal{M}'$ and Under. Recall that $x_{ij}(v_i)=\sum_{v_{-i}\sim T_{-i}}D_{-i}(v_{-i})x_{ij}(v_i,v_{-i})$. Thus Under can be rearranged as follows.

$$\begin{aligned}&\text{Under}\\ =&\sum_{i\in N}\sum_{v_i\in T_i}\sum_{j\in M}D_i(v_i)x_{ij}(v_i)\\ &\quad\cdot\sum_{t_{-i}\in T_{-i}}D_{-i}(t_{-i})(v_{ij}-c_j)\cdot\mathbb{I}[v_{ij}<\beta_{ij}(t_{-i})]\\ =&\sum_{i\in N}\sum_{v\in T}\sum_{j\in M}D(v)x_{ij}(v)\sum_{t\in T}D(t)(v_{ij}-c_j)\\ &\quad\cdot\mathbb{I}[v_{ij}<\beta_{ij}(t_{-i})]\\ =&\sum_{v\in T}D(v)(\sum_{i\in N}\sum_{t\in T}D(t)\sum_{j\in M}x_{ij}(v)(v_{ij}-c_j)\\ &\quad\cdot\mathbb{I}[v_{ij}<\beta_{ij}(t_{-i})]).\end{aligned}$$

It is easy to see that Under is maximized by some deterministic allocation. So we focus on a deterministic allocation rule x.

Let us consider the innermost summation. For any item j, denote by i the buyer such that $x_{ij}(v)=1$ and $\mathbb{I}[v_{ij}<\beta_{ij}(t_{-i})]=1$. Then v_{ij} is one of the reserve prices for item j in $\mathcal{M}'$. Next we see how much profit does mechanism $\mathcal{M}'$ make from selling item j.

If $c_j\geq P_{ij}(t_{-i})$, $\mathbb{I}[v_{ij}<\beta_{ij}(t_{-i})]=1$ implies $v_{ij}<c_j$. In $\mathcal{M}'$, item j will be either sold with price at least c_j or unsold, which means the profit is always nonnegative from selling item j and naturally greater than $v_{ij}-c_j<0$.

If $c_j<P_{ij}(t_{-i})$, $\mathbb{I}[v_{ij}<\beta_{ij}(t_{-i})]=1$ implies $v_{ij}<P_{ij}(t_{-i})$. Note that the highest bidder for item j (denoted by (i_j,j)) must have value at least $P_{ij}(t_{-i})$. Since $P_{ij}(t_{-i})>v_{ij}$ and $P_{ij}(t_{-i})>c_j$, (i_j,j) is willing to take item j at price $\max\{v_{ij},c_j,P_{i_jj}(t_{-i_j})\}$ which is at least v_{ij}. Then the profit is at least $v_{ij}-c_j$.

Combining the above two cases,

$$\begin{aligned}PFT(\mathcal{M}';\hat{\mathcal{I}}^c)\quad\geq&\sum_{i\in N}\sum_{t\in T}D(t)\sum_{j\in M}x_{ij}(v)(v_{ij}-c_j)\\ &\quad\cdot\mathbb{I}[v_{ij}<\beta_{ij}(t_{-i})].\end{aligned}$$

Therefore,

$$\text{Under}\leq\sum_{v\in T}D(v)PFT(\mathcal{M}';\hat{\mathcal{I}}^c)\leq OPT(\hat{\mathcal{I}}^c),$$

which finishes the proof of Lemma A.3. □

Over is also upper-bounded by $OPT(\hat{\mathcal{I}}^c)$, and the proof is almost the same with Lemma 14 of [5], thus omitted here.

Lemma A.4. Over $\leq OPT(\hat{\mathcal{I}}^c)$.

Next, we use $PFT(\mathcal{M}_{1LA};\mathcal{I}^c)$ and $PFT(\mathcal{M}_{BVCG};\mathcal{I}^c)$ to bound Tail and Core. Recall that $PFT(\mathcal{M}_{1LA};\mathcal{I}^c)=r\leq OPT(\hat{\mathcal{I}}^c)=PFT(\mathcal{M}_{IT};\mathcal{I}^c)$.

Lemma A.5. Tail $\leq r$.

Proof.

$$\begin{aligned}&\text{Tail}\\ \leq&\sum_{i\in N}\sum_{j\in M}\sum_{t_{-i}\in T_{-i}}D_{-i}(t_{-i})\sum_{v_{ij}>\beta_{ij}(t_{-i})+r_i(t_{-i})}D_{ij}(v_{ij})\\ &\quad\cdot(v_{ij}-\beta_{ij}(t_{-i})+\beta_{ik}(t_{-i})-c_k)\\ &\quad\cdot\Pr_{v_{i,-j}\sim D^B_{i,-j}}[\exists k\neq j, v_{ik}-\beta_{ik}(t_{-i})\\ &\qquad\geq v_{ij}-\beta_{ij}(t_{-i})]\\ \leq&\sum_{i\in N}\sum_{j\in M}\sum_{t_{-i}\in T_{-i}}D_{-i}(t_{-i})\sum_{v_{ij}>\beta_{ij}(t_{-i})+r_i(t_{-i})}D_{ij}(v_{ij})\\ &\quad\cdot(v_{ij}-\beta_{ij}(t_{-i})+\beta_{ik}(t_{-i})-c_k)\\ &\quad\cdot\sum_{k\neq j}\Pr_{v_{ik}\sim D^B_{ik}}[v_{ik}-\beta_{ik}(t_{-i})\geq v_{ij}-\beta_{ij}(t_{-i})]\\ \leq&\sum_{i\in N}\sum_{j\in M}\sum_{t_{-i}\in T_{-i}}D_{-i}(t_{-i})\\ &\quad\cdot\sum_{v_{ij}>\beta_{ij}(t_{-i})+r_i(t_{-i})}D_{ij}(v_{ij})\sum_{k\neq j}r_{ik}(t_{-i})\\ \leq&\sum_{i\in N}\sum_{j\in M}\sum_{t_{-i}\in T_{-i}}D_{-i}(t_{-i})r_i(t_{-i})\cdot\sum_{v_{ij}>\beta_{ij}(t_{-i})+r_i(t_{-i})}D_{ij}(v_{ij})\end{aligned}$$

$$\begin{aligned}
&\leq \sum_{i\in N}\sum_{j\in M}\sum_{t_{-i}\in T_{-i}} D_{-i}(t_{-i})(r_i(t_{-i})+\beta_{ij}(t_{-i})-c_j) \\
&\qquad \cdot \sum_{v_{ij}>\beta_{ij}(t_{-i})+r_i(t_{-i})} D_{ij}(v_{ij}) \\
&\leq \sum_{i\in N}\sum_{j\in M}\sum_{t_{-i}\in T_{-i}} D_{-i}(t_{-i})r_{ij}(t_{-i}) = r.
\end{aligned}$$

The first and fifth inequalities are because $\beta_{ik}(t_{-i}) \geq c_k$ for all $k \in M$. The second inequality is by union bound. The third and sixth inequalities hold by the definition of $r_{ik}(t_{-i})$. □

Lemma A.6. Core $\leq 2r + 2PFT(\mathcal{M}_{BVCG};\mathcal{I}^c)$.

Proof. For the random variable $v_{ij} \sim D_{ij}$, define the following two new random variables: $b_{ij}(t_{-i}) = (v_{ij} - \beta_{ij}(t_{-i}))\mathbb{I}[v_{ij} \geq \beta_{ij}(t_{-i})]$ and $d_{ij}(t_{-i}) = b_{ij}(t_{-i})\mathbb{I}[b_{ij}(t_{-i}) \leq r_i(t_{-i})]$. Note that $\sum_j b_{ij}(t_{-i})$ is buyer i' utility in the VCG mechanism with reserve price c. Then

$$\text{Core} = \sum_{i\in N}\sum_{j\in M}\sum_{t_{-i}\in T_{-i}} D_{-i}(t_{-i}) \mathop{\mathbb{E}}_{v_i\sim D_i^B} d_{ij}(t_{-i}).$$

Set $\hat{e}_i(t_{-i}) = \sum_{j\in M} \mathop{\mathbb{E}}_{v_i\sim D_i^B}[d_{ij}(t_{-i})] - 2r_i(t_{-i})$. By Lemma 12 in [5],

$$\Pr[\sum_{j\in M} d_{ij}(t_{-i}) \leq \hat{e}_i(t_{-i})] \leq \frac{1}{2}.$$

Thus

$$\Pr[\sum_{j\in M} b_{ij}(t_{-i}) \geq \hat{e}_i(t_{-i})] \geq \Pr[\sum_{j\in M} d_{ij}(t_{-i}) \geq \hat{e}_i(t_{-i})] \geq \frac{1}{2}.$$

Note that the entry fee $e_i(v_{-i})$ is the median of the random variable $\sum_{j\in M}(v_{ij}-\beta_{ij}(v_{-i}))^+$, then $e_i(v_{-i}) \geq \hat{e}_i(v_{-i})$ and $\Pr[\sum_{j\in M} b_{ij}(t_{-i}) \geq e_i(t_{-i})] = \frac{1}{2}$. Therefore,

$$\begin{aligned}
PFT(\mathcal{M}_{BVCG};\mathcal{I}^c) &\geq \frac{1}{2}\sum_{i\in N}\sum_{t_{-i}\in T_{-i}} D_{-i}(t_{-i})\hat{e}_i(t_{-i}) \\
&= \frac{\text{Core}}{2} - r.
\end{aligned}$$

□

Finally, we obtain the main theorem of this section.

Theorem A.2. *When the buyers have additive valuations, Mechanism $\mathcal{M}_A$ is DSIC and is an 8-approximation to the optimal profit for production-cost markets.*

Proof. Combining Lemma A.1 with Lemmas A.2, A.3, A.4, A.5 and A.6, for any BIC mechanism $\mathcal{M} = (x,p)$ for production-cost markets and any production-cost market instance $\mathcal{I}^c = (N,M,D,c)$,

$$PFT(\mathcal{M};\mathcal{I}^c) \leq 6PFT(\mathcal{M}_{IT};\mathcal{I}^c) + 2PFT(\mathcal{M}_{BVCG};\mathcal{I}^c).$$

By selling items using $\mathcal{M}_{IT}$ with probability $\frac{3}{4}$ and using $\mathcal{M}_{BVCG}$ with probability $\frac{1}{4}$, we have an 8-approximation to the optimal profit. □