These Polar Twins: Opinion Dynamics of Intervals

Amr Alhossary
Nanyang Technological University
Singapore
aalhossary@ntu.edu.sg

Alan Tsang
National University of Singapore
Singapore
akhtsang@nus.edu.sg

Zinovi Rabinovich
Nanyang Technological University
Singapore
zinovi@ntu.edu.sg

Svetlana Obraztsova
Nanyang Technological University
Singapore
lana@ntu.edu.sg

Judy Goldsmith
University of Kentucky
Lexington, Kentucky, USA
goldsmit@cs.uky.edu

Alex Wyllie
University of Kentucky
Lexington, Kentucky, USA
wylliealex0@gmail.com

ABSTRACT

A common assumption in opinion dynamics is that an individual's opinion is an atomic point in the opinion space, or a central point accompanied by a certain degree of uncertainty or indifference. While this is satisfactory in many domains, we propose an extension where an individual's preference is an interval in the space of opinions with two characteristic endpoints. These endpoints represent extreme expressions of the same opinion and can be influenced by different sources. For instance, an individual's political opinion, as seen through the lense of the expressed range of political stances, may be influenced by their family on one endpoint, and by their peers on the other endpoint. In this paper, we introduce a general model for capturing this type of interval opinion dynamics and examine several empirical features of this model in the presence of polarizing extremists.

KEYWORDS

Opinion Dynamics; Interval Opinions

1 INTRODUCTION

The field of opinion dynamics examines the spread of ideas or concepts throughout a population. In recent years, it has become a topic of particular interest to the multiagent systems community, and the wider AI research community. This interest has been driven, in no small part, by the rapid spread of news and rumors, and the polarization of individual opinions enabled by persistent and ubiquitous social media interactions. The study of opinion dynamics is instrumental in understanding how these interactions help shape public discourse and discussions.

Early studies of opinion dynamics focused on modeling the spread of new ideas introduced to a population. In these models of innovation diffusion, individuals chose whether to adopt a novel idea based on the actions of their neighbors, by way of a repeated coordination game. In particular, a new idea was adopted when more than a threshold number of neighbors have already adopted it. This model was used to examine the uptake of new technologies such as antibiotics by physicians [13] and hybrid corn by farmers [23], but can, in the modern context, be easily adapted to examine the competition between smart device operation systems and news media platforms.

In innovation diffusion models, the opinions of individuals are binary variables — either the novel idea is adopted or not. In opinion dynamics, opinions take on real values within a fixed interval. This allows the model to capture a gradual shift of opinions, where repeated interactions between individuals cause their opinions to drift closer together. The model may incorporate a value for individuals' uncertainty [8, 11, 12, 14, 25], or may choose to capture this uncertainty as an interval in the opinion space [18, 28].

In certain domains, the actual opinion held by an individual may *itself* be an interval, modeling not a range of uncertainty but a specific set of feasible values, embedding in a high dimensional opinion space. Such scenarios arise naturally. For example, in the realm of political discourse, an individual may accept a particular combination of values influenced by their peers, and a second combination of values passed down through their family; naturally, between these two extremal points lies an interval of feasible values representing compromises between those two ideals. As individuals interact, the interval of feasible values will shift. Since the interval does not merely represent uncertainty or indifference, the two endpoints are not necessarily interchangeable and may interact with the community in different ways. Like the eponymous "Polar Twins", Jekyll and Hyde, these endpoints are influenced by different associates in the social network, defined by the different natural of individual relationships.

In this paper, we propose an opinion dynamics model of interval-opinions, that capture sets of feasible options preferred by the agents. Our model allows the interval-opinion endpoints to interact asymmetrically within a social network. The paper's layout is as follows: We begin with a related work overview in Section 2, followed by the definition of our model in Section 3. Sections 4 and 5 present our empirical studies and results. Section 6 concludes the paper.

2 RELATED WORK

Recent work in opinion dynamics incorporates the notion of uncertainty alongside a real number representation of an agent's preference. Hegselmann and Krause [14] were the first to propose such a model, termed the bounded confidence model. In this model, agents' numerical preferences grow closer to their neighbors through repeated interactions via weighted averaging. However, agents are selective in their interactions based on their level of uncertainty: confident (low uncertainty) agents only interact with agents whose

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opinions are similar to theirs, while uncertain agents are less discriminating. The behavior of this model is well understood (see [17] and [20], for instance), known to converge in polynomial time [3], and has been extended into a more general framework termed *Diffusive Influence Systems* [7].

Carvalho and Larson [6] adopt a similar approach. They model a scenario of experts conferring and adjusting their opinions according to each other, giving a higher weight to similar opinions. They show that this dynamic follows naturally if utilities follow a quadratic scoring function, and is guaranteed to converge to a stable configuration.

The Hegselmann-Krause model has been refined and used to study polarization of opinions in communities [9, 24]. While our experiments explore similar polarization scenarios, our model is fundamentally different from the classic Hegselmann-Krause model in that opinions are modelled as intervals in high dimensional space, and a rich class of interaction dynamics are possible beyond weighted averaging.

Finally, we must mention the use of interval valued fuzzy sets to extend the Hegselmann-Krause bounded confidence opinion dynamic models in different ways by Wang and Mendel [26], and Gasparri and Oliva [11]. Our work differ in two significant ways. The fuzzy set models represent uncertainty of a point-preference in the opinion space; this is fundamentally different from intervals in our model, which is the true preference of the individual, representing a set of feasible points. Moreover, our model allows a richer class of interactions between individuals, including the allowance for the endpoints of the intervals to be influenced asymmetrically.

Other researchers in the multiagent community have also explored the diffusion of non-numerical opinions in networks. Brill et al. [5] examined the interaction dynamics of rank ordered preferences. Each agents' preference is a total ordering of the candidates. In each round, an agent selects a random pair of candidates, and polls her neighbors for their respective ranking between the two candidates, swapping her own ranking of them to match the majority result. They show that this *Pairwise Preference Diffusion* is complex, and find termination and convergence conditions for only certain types of graphs.

3 MODEL

We begin our research by introducing the simplest variation of non-atomic opinions: intervals. For convenience, we view this situation as a pair of coupled extremes, termed "Castor" and "Pollux", and an *interval opinion* is any combination of these extremes. As this is the first study of *interval opinions*, we adopt the view of the C&L [6] model, and assume that extremes are points in a simplex $\mathbb{S}^{d-1} \subset \mathbb{R}^d$. This allows any portion of the interval opinion to be interpreted as a preference over d issues, e.g. in politics. Again, the fact that the opinion interval groups mutliple such preference points is *not* to be taken as indiference between them or uncertainty regarding the preference. Rather, interval opinions should be taken as a stand-alone concept.

Formally, in this paper, we will consider *interval opinions* of the form $[c, p]$ (a segment in space), where $c, p \in \mathbb{S}^{d-1} \subset \mathbb{R}^d$. A population of n opinions $\{[c_i, p_i]\}_{i\in[n]}$, where $[n] = \{1, 2, \ldots, n\}$, will be organised into a *joint state* matrix $\mathbf{x} = [c_1, \ldots, c_n, p_1, \ldots, p_n] \in M_{d\times(2n)}(\mathbb{R})$. We will term c_i and p_i, respectively, Castor-i and Pollux-i. For easier verbal expression we will refer to the interval opinion $[c_i, p_i]$ as Castor-i-Pollux, or C-i-P for short. We will want to study various forms of mutual influence witin a population of interval opinions and the development of the joint state in time, i.e. interval opinion dynamics. To begin our study, in this paper, we will concentrate on linear dynamics. That is, whatever influences an interval-opinion $[c, p]$ experiences, absorbing them will result in the same linear transformation applied to all constituent points of the interval-opinion. Hence, all linear dynamics will have the form: $x^{t+1} = x^t\mathbb{D}[x^t]$, where $\mathbb{D}[x^t]$ is a $2n \times 2n$ column-stochastic *dynamics matrix*, parameterised by and dependent on the joint state. Where the joint state is clear from the context, we will simply write $\mathbb{D}$. Due to the linear nature of opinion transformation by the dynamic, we can break down the dynamics matrix into the following sub-structure:

$$\mathbb{D} = \left(\begin{array}{c|c} \mathbb{D}^{c\to c} & \mathbb{D}^{c\to p} \\ \hline \mathbb{D}^{p\to c} & \mathbb{D}^{p\to p} \end{array}\right),$$

where $\mathbb{D}^{c\to c}, \mathbb{D}^{c\to p}, \mathbb{D}^{p\to c}, \mathbb{D}^{p\to p}$ are matrices of size $n \times n$ with non-negative elements, and for all $j \in [n]$ holds

$$\sum_{i\in[n]} \left(\mathbb{D}^{c\to c}_{ij} + \mathbb{D}^{p\to c}_{ij}\right) = \sum_{i\in[n]} \left(\mathbb{D}^{c\to p}_{ij} + \mathbb{D}^{p\to p}_{ij}\right) = 1.$$

Notice how the sub-structure notation underlines the use of linear dynamics to define the influence of interval-opinions on each other in terms of coupled influence of their end-points. In particular, $\mathbb{D}^{c\to p}$, for example, captures how the Castors of the joint state $\mathbf{x}^t$ influence the Pollucis of the joint state $\mathbf{x}^{t+1}$.

In the above notation, the C&L model [6] would stipulate that $\mathbb{D}_{ij} \propto \frac{1}{\epsilon + d(\eta_i\|\eta_j)}$, where $d(\cdot\|\cdot) : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}_{\geq 0}$ is some distance function, $\eta_i = \begin{cases} c_i & i \in [n] \\ p_{i-n} & i \in [2n], i > n \end{cases}$ and similarly for η_j. The C&L model was particularly interested in the norm $d\left(\eta_i\|\eta_j\right) = \sqrt{\frac{\sum_{k=1}^{d}(\eta_{ik} - \eta_{jk})^2}{d}}$. We adopt this distance function as well.

It must be noted that the C&L model describes the behaviour of single-point atomic opinions that are fully linked (every point influences every other). As a result, it would treat the joint state $\mathbf{x}$ as a set of $2n$ opinions, without any regard to the Castor-Pollux relationship. Setting $\mathbb{D}_{ij} = d\left(\eta_i\|\eta_j\right)$ for $i, j \in [2n]$ fully reproduces that effect in our notation.

However, this does not demonstrate the effects of the interval-opinion concept. To that end, we will need to study more complex dynamics that: a) take into account the interval as an object; b) subject the extremes of the interval to conflicting influences. We, therefore, formulate the following sequence of dynamics with incresing involvement of the interval nature of the opinion.

Base-line Dynamics: Independent Castor and Pollux (ICaP). As a first step, we assume that interval opinion extremes form two independent sets, each with its interval influence structure. E.g., a teenager's opinion of what's "cool"is an attempt to reconcile extreme conformity with extreme individuality, with each extremity being subject to independent discussion, fashion and influence. In our formalism, this can easily be captured by the following constraints on the linear dynamics $\mathbb{D}$:

$$\mathbb{D}^{p\to c} = \mathbb{D}^{c\to p} = 0_{n\times n}$$

$$\mathbb{D}^{c\to c}_{ij} \propto \frac{1+\delta_{ij}\epsilon_{\Upsilon\omega}}{\epsilon + d\left(c_i \| c_j\right)},\ and\ \mathbb{D}^{p\to p}_{ij} \propto \frac{1+\delta_{ij}\epsilon_{\Upsilon\omega}}{\epsilon + d\left(p_i \| p_j\right)}$$

where $i, j \in [n]$, and δ_{ij} is the standard Kronecker delta function, so that $\delta_{ij} = 1$ if an only if $i = j$, and $\epsilon_{\Upsilon\omega}$ is the self-inertia parameter. Notice that $\epsilon_{\Upsilon\omega}$ will influence the entire interval, not just the exreme points.

Independent Network Castor and Pollux (INCaP). Of course teenagers are also subject to strict grouping and socialisation limitations, and their opinion will not be freely influenced by all. Rather, a complex network of influences is formed, and (conflicted as teenagers are) each opinion extreme can have an independent influence network.

In our formalism this is expressed by Castores (Pollucis) not fully influencing each other. Instead, their influence network is captured by two directed graphs, G^c and G^p, over a node set $[n]$ that govern mutual influences of opinion extremes. In particular, if an edge $(i \to j) \in G^c$, then the opinion of Castor-i will influence Castor-j. Similarly, if an adge $(i \to j) \in G^p$, then the opinion of Pollux-i will influence Pollux-j. We will assume that all self edges are always present, i.e. $(i \to i) \in G^c$ and $(i \to i) \in G^p$. We will further denote by $pa(j, G)$ all nodes that can influence node j within graph G. Formally,

$$pa(j, G) = \{i \in [n] | (i \to j) \in G\}.$$

Then, given influence graphs G^c and G^p, we limit the opinion dynamics so that the following holds for all $i, j \in [n]$:

$$\mathbb{D}^{p\to c} = \mathbb{D}^{c\to p} = 0_{n\times n}$$
$$\mathbb{D}^{c\to c}_{ij} \propto \frac{\delta^{[G^c]}_{ij} + \delta_{ij}\epsilon_{\Upsilon\omega}}{\epsilon + d\left(c_i \| c_j\right)} \quad and \quad \mathbb{D}^{p\to p}_{ij} \propto \frac{\delta^{[G^p]}_{ij} + \delta_{ij}\epsilon_{\Upsilon\omega}}{\epsilon + d\left(p_i \| p_j\right)}$$

where $\delta^{[G]}_{ij} = \begin{cases} 1 & (i \to j) \in G \\ 0 & o/w \end{cases}$ is a Kronecker indicator.

Coupled Network Castor and Pollux (CoNCaP). Continuing with our teenage "cool" opinion, we must recognise that as teenagers grow they do attempt to reconcile their "coolness" opinion extremes. As a result, these extremes will not developed independently, as the previous two variations of our model imply. Rather, a certain coupling and attraction between the extremes will appear. In our formalism, this means that Castor-i and Pollux-i will be co-dependent in some sense. To capture this, we allow diagonals of dynamics sub-matrices $\mathbb{D}^{c\to p}$ and $\mathbb{D}^{p\to c}$ to vary.

We will distinguish between two variations of this constraint: a persistent CoNCaP-ϕ, where the Castor-Pollux influence is fixed; and the dynamic CoNCaP-β, where the influence strength develops similarly to the influence withing the sets of Castores and Pullucis, though may be weighted.

CoNCaP-ϕ: Here, we set a single parameter $\phi \in (0, 1)$, and limit the dynamics $\mathbb{D}[x]$ as follows for all $i, j \in [n]$:

$$\begin{aligned} \mathbb{D}^{p\to c}_{ij} = \mathbb{D}^{c\to p} &= \phi\delta_{ij} \\ \mathbb{D}^{c\to c}_{ij} &\propto \frac{(1-\phi)\left(\delta^{[G^c]}_{ij} + \delta_{ij}\epsilon_{\Upsilon\omega}\right)}{\epsilon + d\left(c_i \| c_j\right)} \\ \mathbb{D}^{p\to p}_{ij} &\propto \frac{(1-\phi)\left(\delta^{[G^p]}_{ij} + \delta_{ij}\epsilon_{\Upsilon\omega}\right)}{\epsilon + d\left(p_i \| p_j\right)} \end{aligned}$$

CoNCaP-β: Although we will retain the ability to weigh the influence via the $\beta \in (0, 1)$ parameter, the mutual influence in a C-i-P pair will no longer be fixed. Rather, it will vary and go through the same normalisation process as any other influence. Thus, the dynamics are described by the following constraints for all $i, j \in [n]$:

$$\begin{aligned} \mathbb{D}^{p\to c}_{ij} = \mathbb{D}^{c\to p} &\propto \frac{\beta\delta_{ij}}{\epsilon + d\left(c_i \| p_j\right)} \\ \mathbb{D}^{c\to c}_{ij} &\propto \frac{(1-\beta)\left(\delta^{[G^c]}_{ij} + \delta_{ij}\epsilon_{\Upsilon\omega}\right)}{\epsilon + d\left(c_i \| c_j\right)} \\ \mathbb{D}^{p\to p}_{ij} &\propto \frac{(1-\beta)\left(\delta^{[G^p]}_{ij} + \delta_{ij}\epsilon_{\Upsilon\omega}\right)}{\epsilon + d\left(p_i \| p_j\right)} \end{aligned}$$

where δ_{ij} is the standard Kronecker delta function, so that $\delta_{ij} = 1$ if and only if $i = j$.

Fully Coupled Network Castor and Pollux (FCoNCaP). Stepping one more step towards acknowledging that Castor-i and Pollux-i are none other than two extremes of the same interval opinion, we explore a dynamic where the coupling is taken into account within Castor (Pollux) social network, G^c (respectively, G^p), influence calculation. Namely, if $(i \to j)$ is an edge in the influence graph G^c (respectively, G^p), then the opinion of c_j (respectively, p_j) will be adjusted towards the entire **interval** of opinions between c_i and p_i.

Intuitively, this is a teenager's attempt to adapt her opinion to the entire expression range of her idol, following both the on-stage and (the apparent) daily life-style. Formally, let us define *proximity opinion points* y^p_{ij} and y^c_{ij} as follows:

$$\begin{aligned} \alpha^c_{ij} &= \arg\min_{\alpha\in[0,1]} d\left(\alpha c_i + (1-\alpha)p_i \| c_j\right) \\ \alpha^p_{ij} &= \arg\min_{\alpha\in[0,1]} d\left(\alpha p_i + (1-\alpha)c_i \| p_j\right) \\ y^c_{ij} &= \alpha^c_{ij}c_i + (1-\alpha^c_{ij})p_i \\ y^p_{ij} &= \alpha^p_{ij}p_i + (1-\alpha^p_{ij})c_i \end{aligned}$$

Figure 1 shows how proximity points are calculated, when C-i-P influences C-j-P. Notice that calculating most proximal point y^p_{ij} results essentially in projecting p_j onto the line that passes through c_i and p_i. At the same time, the proximal point y^c_{ij} coinsides with c_i.

Now, extending the CoNCaP-β variation, we rewrite the opinion dynamics constraints to become for all $i, j \in [n]$:

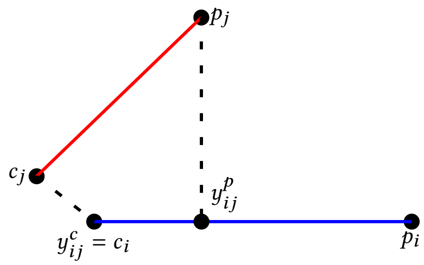


Figure 1: Calculating proximity opinion points y^p_{ij} and y^c_{ij}.

$$\mathbb{D}^{p \to c}_{ij} \quad \propto \quad (1-\delta_{ij}) \frac{(1-\beta)\delta^{[G^c]}_{ij}(1-\alpha^c_{ij})}{\epsilon + d\left(y^c_{ij} \| c_j\right)} + \delta_{ij} \frac{\beta}{\epsilon + d\left(p_i \| c_j\right)}$$

$$\mathbb{D}^{c \to c}_{ij} \quad \propto \quad \frac{(1-\beta)\left(\delta^{[G^c]}_{ij} + \delta_{ij}\epsilon_{\gamma\omega}\right)\alpha^c_{ij}}{\epsilon + d\left(y^c_{ij} \| c_j\right)}$$

with $\mathbb{D}^{c \to p}_{ij}$, $\mathbb{D}^{p \to p}_{ij}$ defined symmetrically to $\mathbb{D}^{p \to c}_{ij}$ and $\mathbb{D}^{c \to c}_{ij}$.

4 EXPERIMENTAL STUDY: SETUP

The range of questions regarding an opinion dynamics commonly includes questions of convergence per se and the properties of the convergence set. The former can be used to simulate social stability, rigidity and consensus [6, 14], while the latter is brought about to model political polarisation, community detection, and stock prices [2, 19, 21]. However, as was mentioned in Section 2, these works concentrate on opinions represented by a (multidimensional) *point*, following a well-developed geometric intution. E.g. community members develop a degree of commonality that is readily captured by clusters their point-opinions. Following the same intuition, political polarisation is conveniently captured by population clustering, and the *degree of polarisation* is readily produced by clustering scores. In particular, if the Silhouette score[22] [1] of such a clustering is concentrated around 1(one), it is understood that the society is extremely polarised and it will be difficult to find a political compromise. Alas, it is much harder to cluster non-atomic entities like intervals. In fact, the standard intuition of a polarised society no longer works in our model. This is also true for other opinion dynamics questions.

Thus, before we proceed to formal model analysis and applications, we first need to rebuild a natural intuition wrt meaning and implications of interval-opinion dynamics outcomes. In this section we build such an intuition by redeveloping polarisation interpretations, and provide some experimental data to support them.

4.1 General Experimental Setup

For each simulation run in our preliminary set of experiments we are generating random interval opinions in $\mathbb{R}^3$ (so that $d = 3$, and $c_i, p_i \in \mathcal{S}^2$ for all $i \in [n]$), and *impose* distinct social connectivity graphs G^c and G^p on Castores and Pollucis respectively. Social connectivity graphs G^c and G^p are a result of a random graph generation deemed well suited for social network studies. In particular, our simulation is capable to employ Barabasi-Albert [1], Erodos-Renyi [4, 10], Watts-Strogatz [27] and Kleinberg's Small World [16] graph generators. However, for our initial experimental data set we chose to concentrate on Barabasi-Albert graphs.

Simulation runs are grouped into series characterised by settings of model parameters and the number of Castore-Pollux pairs. Each series includes at least 50 runs – i.e. each setting of model parameters is tested for 50 times with different initial joint state $\mathbf{x}$ and social grphs G^c and G^p. Where possible we ensure that we can match parameter characteristics across series that belong to different dynamic models. Behavour of each series is automatically analysed using a set of measures, as described in the next sub-section. Comparative statistics are then drawn, and we interpret those comparisons as the difference in behavioural patterns between opinion dynamic models.

4.2 Polarisation Challenge

It is of current interest to study whether, and when, the introduction of persistent or prolonged opinion injection into a social network has a polarisation effect. That is, the difference in opinions of various network participants increases, and sub-communities gravitate towards the injected opinion sources. To study the effects of persistent opinion injection (aka polarisation), we introduce a secondary (randomised) augmentation into the social graphs of Castores and Pollucis. Specifically, we create a fixed percentage of strictly influencing (only outgoing edges), *"stubborn"* opinions. We then polarise/extremise the stubborn opinions by shifting them towards distinct and distant reference points. This effectively creates a prolonged ijection of those reference point opinions into the network. Opinion dynamics simulation then reveals the effect of this injection.

We proceed as follows. Given an initial joint configuration $\mathbf{x}$, its associated social graphs G^c and G^p, and two polar locations $P^+ = (1.0, 0.0, 0.0)$ and $P^- = (0.0, 0.5, 0.5)$, we introduce the following randomised modifications (and their parameters) before we simulate the dynamics:

Opinion Injection Ratio ρ: Ensure that both G^p and G^c have $\rho \in [0, 1]$ proportion of nodes that have only outgoing edges (they influence, but do not get influenced back) – termed *stubborn extremes*. If a graph does not contain a sufficient number of stubborn extremes, select additional nodes at random and invert their social connectivity to make them stubborn; If a graph contains more than a given ratio of stubborn extremes, select a necessary portion of them at random and randomise the social connectivity direction of the selected nodes (to "de-stubborn" them).

Polarisation Coefficient η: Shift the *stubborn opinions*, i.e. those with at least one end of the interval-opinion being a stubborn extreme, towards one of the poles (chosen at random) W.l.o.g. let's assume that the stubborn extreme is c_i, then it is modifed to either $\eta c_i + (1-\eta)P^+$ or $\eta c_i + (1 - \eta)P^-$

The polarising shift of stubborn opinions comes in two flavours: a) Coherent Extremisation, where both ends of a stubborn interval-opinion are augmented towards the same pole; b) (Simple) Polar Extremisation, where only the stubborn extreme is shifted, and both interval-opinion ends are stubborn, then they are shifted independently.

[1] Commonly used in machine learning, the Silhouette score of a point a measures the ratio between the average distance between a and other points within a cluster to the best average distance between a and points of another cluster.

4.3 Measuring the effects of polarisation

In a standard situation, where any opinion is represented by a single point and no coupling occurs, polarisation has a detrimental effect that is largely well intuited and understood. E.g. if we say that an electorate set of opinions is highly polarised, we would generally mean that it is easily clustered into tight groups around distinct candidate opinions. In clustering terms, we would say that the distribution of the Silhouette score is highly concentrated around 1. We will then intuitively understand that a voter would be anxious about the elections' outcome as any, but a very specific candidate, would be seen as a Doom's Day for that voter.

However, if opinions are represented by intervals, the intuition of point-based polarisation and the Silhouete score expressing overall anxiety with election outcomes does not apply any longer. This is because the size of the opinion interval matters, and not just the distance between voter and candidate opinions. Consider the simplest situation where a voter's interval opinion and candidate point opinions lay on the same line, and inspect the variety of possibilities in Figure 2. It is easy to see that the anxiety would grow in subsequences of example intervals 1-4, 5-11 and 12-15.

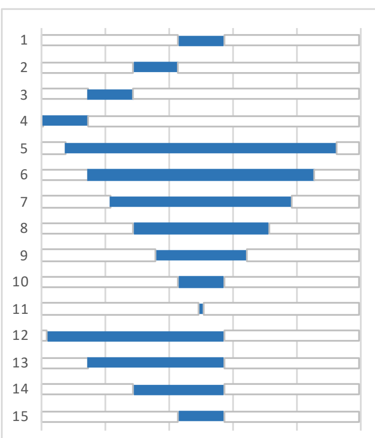


Figure 2: Intuition for *anxiety*, $\underline{\nabla}$: examples of interval opinion between two candiate extremes.

We need to capture how *anxious* each voter is with the difference between the best and the worst candidate from her point of view. Voters 5-11 in Figure 2 clearly demonstrate the impact of interval size. Both options are equally distant from the interval, so a voter would be indifferent between either candiate prevailing. However, we intuitively understand that voter-5 experiences far less anxiety than voter-9. On the other hand, voter-11 may be simply inundated by the distance to any candidate to even care, so the anxiety level may drop for him. While several measurements that would correctly capture these examples are possible, we will be taking a leaf from [15], and concentrate on entropy-based measurements. Recall, the two basic entropic functions are $H(x_1, \ldots, x_n) = -\sum_{i=1}^{n} x_i \log x_i$ and $h(x_1, \ldots, x_n) = H(\frac{x_1}{x}, \ldots, \frac{x_n}{x})$, where $x = \sum_{i=1}^{n} x_i$ and $x_i \geq 0$[2].

Let O denote the set of all C-i-P opinions, where either c_i or p_i is *stubborn* in their respective social graphs. Denote v a C-j-P opinion that is not in O, and consider two possible metrics for the distance between intervals $o \in O$ and $v \notin O$: $d_1(o\|v) = \min\{d(y^c_{ij}, c_j), d(y^p_{ij}, p_j)\}$; and $d_2(o\|v) = \max\{d(y^c_{ij}, c_j), d(y^p_{ij}, p_j)\}$. Based on these metrics we define the following two types of opinion discrepancies.

External discrepancies: $s^{\min}(v) = \min_{o\in O} d_1(o\|v)$, $l^{\min} = \max_{o\in O} d_1(o\|v)$, and $l^{\max} = \max_{o\in O} d_2(o\|v)$.

Internal opinion discrepancies: $s^c_* = \min_{o\in O} d(y^c_{ij}, c_j)$, and $s^p_* = \min_{o\in O} d(y^p_{ij}, p_j)$. In addition, denote by $m = d(c_j, p_j)$ the standard distance between the endpoints of the C-j-P opinion. We will investigate the following functions as measure of voter anxiety, $\underline{\nabla}(v)$, in a polarised society.

Internal Anxiety that captures the tension between the two endpoints an interval-opinion as they are subjected to different social influences:

$$\underline{\nabla}^{h,*} = h(s^c_*, m, s^p_*)$$

External Anxiety that captures the discordance between the most and the least attractive subborn opinion:

$$\underline{\nabla}^{h,\min} = h\left(s^{\min}, m(v), l^{\min}\right)$$
$$\underline{\nabla}^{h,\max} = h\left(s^{\min}, m(v), l^{\max}\right)$$

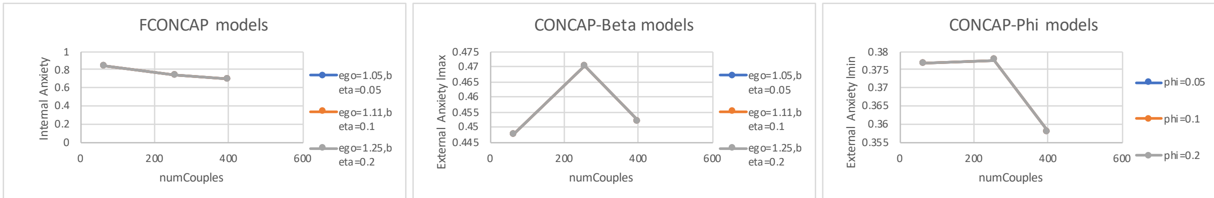
Notice how all expressions above handle dependency of the anxiety on how relaxed is the C-j-P opinion, as expressed by the size of its interval. Finally, we define the expected (population) anxiety $\mathbb{E}\left[\underline{\nabla}\right] = \frac{1}{n-|O|} \sum_{v\notin O} \underline{\nabla}(v)$.

5 EXPERIMENTAL STUDY: RESULTS

First, let us address the issue of the chosen anxiety and contentment measures being appropriate. We begin by noticing that the entropy expression of $\underline{\nabla}^{h,*}$, $\underline{\nabla}^{h,\min}$ and $\underline{\nabla}^{h,\max}$ reflect well the intuition of Figure 2. Now, this by itself is not enough, and it is necessary to ensure that the measure is actually influenced by the relevant parameters of a model. ANOVA analysis of our experimental data showed that our entropy-based anxiety expression strongly correlates with social and personal influence parameters ϕ, β and $\epsilon_{\gamma\omega}$ with extremely low p-values. Entropic anxiety does not correlate with the initial polarisation coefficient η, or (and this is quite surprising) the opinion injection ratio ρ. In other words, entropic anxiety reflects **only** the innate tendency of opinion dynamics to create tension of opinion discrepancies, but it **does not** depend on how aggresively polar opinions are injected into a society. Only the end-effect, how well the polar opinions take hold, is captured by the entropic anxiety.

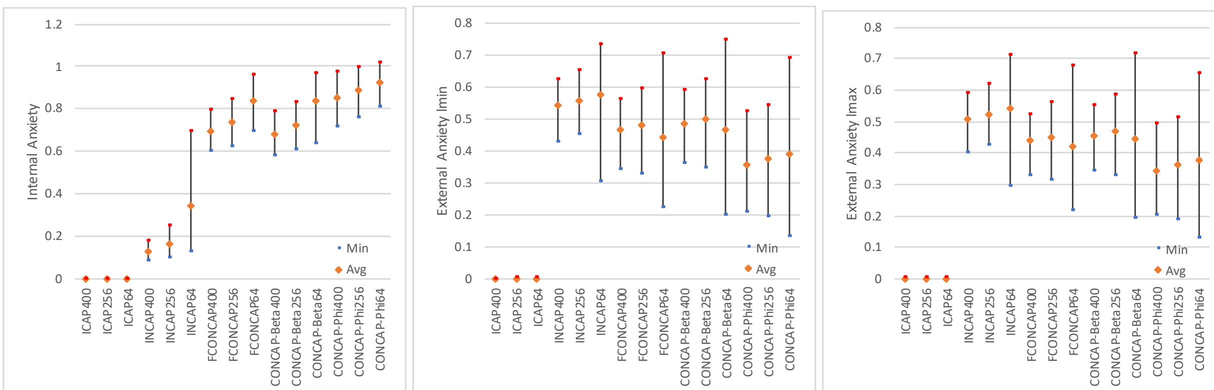
Now, having established our measurement technique as a well tuned tool, we can study the effects of various model parameters on the expected (social) anxiety. One of the key questions is whether the size of a society makes it easier or harder to polarise it. We have ran a series of experiments with fixed social graph generator parameters and increasing number of participating opinions (n=64, 256, 400) and measured the expected anxiety at the point of opinion dynamics convergence. We observed that the internal anxiety persistently decreases with the size of the network, independently of other dynamics parameters (see Figure 3a), while the external anxiety initially increases and then drops off (see Figure 3b) for models with active Castor-Pollux links. We interpret this as a confirmation of

[2] We follow the tradition of taking $0 \log 0 = 0$



(a) Internal anxiety drop **(b) External anxiety drop-off**

Figure 3: Anxiety behaviour as a function of the number of opinions



(a) Internal anxiety ranges **(b) External anxiety ranges**

Figure 4: Inducible anxiety ranges for various models

larger societies being suscepitible to greater polarisation. This is because an increased polarisation of interval-opinions is such that: a) both opinion ends will end-up close to the same polar option, hence the persistent drop in $\underline{\nabla}^{h,*}$; b) the size of the opinion-interval shrinks more and more, forming "closed communities" within larger societies – this replicates the natural tendency of $h^{h,\min}$ and $h^{h,\max}$ to drop off after the initial surge with the reduction of m.

Interestingly enough, in models that include Castor-Pollux link moderation, the internal anxiety is higher. However, the range of anxiety levels that various opinion-dynamics can induce remain largly unchanghed (see Figure 4a). This is not the case for the external anxiety, however. There we see clearly that the range of inducable anxieties shrinks towards its mean as the size of the society increases (Figure 4b). In other words, in a larger society it is less likely that you would be subjected to extremely high or low disappointments from, for example, an election process. Rather, your level of anxiety will be guaranteed – this is a known sociological phenomenon: people can either extremely thrive or suffer in small groups, but in a large society the misery is guaranteed.

6 CONCLUSIONS

In this paper we have introduced a new concept of an interval-opinion dynamics, where a single opinion is non-atomic, as opposed to the classical opinion modelling. In fact, the non-atomic nature of the opinion is critical in our model and can not be replicated via the concepts of opinion indifference or uncertainty. To initiate the discussion and the use of interval-opinions we study its behaviour wrt society polarisation under persistent opinion injection. To this end, we develop an interval-based polarisation measure, experimentally show its relevance and its intuitive implications wrt social polarisation.

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